

UNIQUE FIBER SUM DECOMPOSABILITY OF GENUS 2 LEFSCHETZ FIBRATIONS

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ABSTRACT. By applying the lantern relation substitutions to the positive relation of the genus two Lefschetz fibration over S^2 . We show that $K3\#2\overline{\mathbb{CP}}^2$ can be rationally blown down along seven disjoint copies of the configuration C_2 . We compute the Seiberg-Witten invariant of the resulting symplectic 4-manifolds, and show that they are symplectically minimal. We also investigate how these exotic smooth 4-manifolds constructed via lantern relation substitution method are fiber sum decomposable. Furthermore by considering all the possible decompositions for each of our decomposable exotic examples, we will find out that there is a uniquely decomposing genus 2 Lefschetz fibration which is not a self sum of the same fibration up to diffeomorphism on the indecomposable summands.

1. INTRODUCTION

A nice interplay between the algebra and the topology in the Lefschetz fibration of a symplectic 4-manifold is that the topological surgery operation that generates many interesting examples of an exotic smooth 4-manifold can be performed algebraically via monodromy substitution. One of the well understood mapping class group relation in this regard is the lantern relation which corresponds to the surgical operation of rational blowdown which gives us many interesting examples of exotic smooth 4-manifolds [16, 13]. In Endo-Gurtas' pioneering work, after constructing an exotic smooth 4-manifold E homeomorphic but not diffeomorphic to an elliptic fibration on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ in Example 5.3 [13] via the lantern relation substitutions, they pose a problem about whether the exotic smooth 4-manifold E constructed via monodromy substitution is fiber sum decomposable into a nontrivial fiber sum of other Lefschetz fibrations.

Problem 1. [13] Does E decompose into a nontrivial fiber sum of other Lefschetz fibrations? Is E isomorphic to a fiber sum of two copies of Matsumoto's fibration?

As the manifold E is homeomorphic but not diffeomorphic to $E(1)$, whereas an appropriately twisted fiber sum of two copies of Matsumoto's fibration is also homeomorphic but not diffeomorphic to $E(1)$, this is an interesting problem to investigate. While we cannot answer this problem fully we will remark at the end of our article how E has unique genus 2 fiber sum decomposition up to diffeomorphism on the indecomposable summands if E is fiber sum decomposable. (i.e. we will rule out any other possible genus 2 fiber sum decompositions.)

Date: October 7th, 2015.

2010 *Mathematics Subject Classification.* Primary 57R55; Secondary 57R17.

Key words and phrases. symplectic 4-manifold, Lefschetz fibration, mapping class group, lantern relation, rational blowdown, fiber sum decomposability.

In this article, we will improve the construction of the Akhmedov-Park's exotic smooth 4-manifolds [1] where we found six lantern relations to finding seven lantern relations and also show how some of them are fiber sum decomposable. That is we will show how simply connected, minimal symplectic 4-manifolds $X(n)$ for $2 \leq n \leq 7$ homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (21-n)\overline{\mathbb{CP}}^2$ for $2 \leq n \leq 7$ with $b_2^+ = 3$ and symplectic Kodaira dimensions $\kappa^s = 1$ for $n = 2$ and $\kappa^s = 2$ for $3 \leq n \leq 7$ acquired by starting from genus 2 Lefschetz fibration on $K3 \# 2\overline{\mathbb{CP}}^2$ and applying a sequence of seven rational blowdowns via lantern relation substitutions are all fiber sum decomposable for $2 \leq n \leq 6$ into nontrivial fiber sum of other genus 2 Lefschetz fibrations.

Theorem 2 (Decomposability of $X(n)$ for $2 \leq n \leq 6$). *The genus 2 Lefschetz fibrations $X(n)$ for $2 \leq n \leq 6$ are all decomposable into nontrivial fiber sum of other genus 2 Lefschetz fibrations. Namely, $X(2)$ is isomorphic to an untwisted fiber sum of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with Lefschetz fibration on $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Additionally, $X(3), X(4), X(5), X(6)$ are isomorphic to an untwisted fiber sum of Matsumoto fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with $Z(1), Z(2), Z(3), Z(4)$ respectively.*

Here, $Z(m)$ for $1 \leq m \leq 4$ are examples similar to Endo-Gurtas' genus 2 examples in that they are acquired by starting from genus 2 Lefschetz fibration $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ and applying a sequence of four rational blowdowns via lantern relation substitutions.

After showing decomposability, we will show that the one of the decomposable example $X(2)$ which is a minimal exotic symplectic 4-manifold with the homeomorphism type of $3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}}^2$ with $b_2^+ = 3$ and symplectic Kodaira dimension $\kappa^s = 1$ has the unique genus 2 fiber sum decomposition up to diffeomorphism on the indecomposable summands.

Theorem 3 (Unique decomposition of $X(2)$). *The genus 2 Lefschetz fibration $X(2)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (26, 2)$ must decompose under the genus 2 fiber sum having the indecomposable summands of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Each summands are determined up to diffeomorphism.*

Accordingly, we will narrow down all the possible genus 2 fiber sum decompositions of $X(n) = Y(1) \# Y(2)$ for $3 \leq n \leq 6$ examples with $\kappa^s = 2$ by the consideration on the possible n irreducible singular fibers and s reducible singular fibers pair (n, s) for both $Y(1), Y(2)$ where both summands $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations.

2. PRELIMINARIES

For the convenience of the reader we repeat the preliminary definitions and results from [1, 19] mostly without proofs, thus making our exposition self-contained. The list of topics that need to be recalled are the mapping class groups, the Lefschetz fibrations over \mathbb{S}^2 with details on the Matsumoto's genus two fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$, lantern relation substitution and its relationship with the rational blowdown operation, the symplectic Kodaira dimension and the symplectic minimality.

2.1. Mapping Class Groups. Let Σ_g denote a 2-dimensional, closed, oriented, and connected Riemann surface of genus $g > 0$.

Definition 4. Let $\text{Diff}^+(\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \rightarrow \Sigma_g$, and $\text{Diff}_0^+(\Sigma_g)$ be the subgroup of $\text{Diff}^+(\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \rightarrow \Sigma_g$ that are isotopic to the identity. The mapping class group Γ_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g , i.e.,

$$\Gamma_g = \text{Diff}^+(\Sigma_g) / \text{Diff}_0^+(\Sigma_g).$$

Definition 5. Let α be a simple closed curve on Σ_g . A *right handed Dehn twist* t_α about α is the isotopy class of a self-diffeomorphism of Σ_g obtained by cutting the surface Σ_g along α and gluing the ends back after rotating one of the ends 2π to the right.

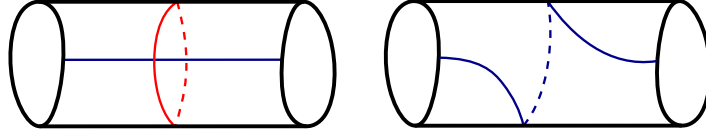


FIGURE 1. A positive Dehn twist to a cylinder about the red curve

The mapping class group Γ_g is finitely generated by $3g - 1$ Dehn twists which was proven by the work of Dehn and Lickorish (cf. [15]). It follows that the conjugate of a Dehn twist is again a Dehn twist. That is, if $f : \Sigma_g \rightarrow \Sigma_g$ is an orientation-preserving diffeomorphism, then it is easy to check that $f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}$.

We will now provide a presentation for the mapping class group of the genus 2 surface Γ_2 . As we will be working mostly with the genus 2 Lefschetz fibrations restricting our attention to Γ_2 will not interfere with the construction we will illustrate.

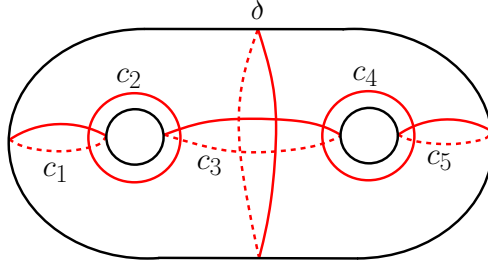
Let t_i ($i = 1, \dots, 5$) be positive Dehn twists along the loops c_i ($i = 1, \dots, 5$) illustrated in Figure 2. The mapping class group Γ_2 of a genus-2 Riemann surface is generated by t_1, \dots, t_5 , and the following relations are defining relations (cf. [5]).

- (1) $t_i t_j = t_j t_i$ if $|i - j| \geq 2$,
- (2) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $i = 1, \dots, 4$,
- (3) $\tau^2 = 1$ where $\tau = t_1 t_2 t_3 t_4 t_5^2 t_4 t_3 t_2 t_1$,
- (4) $(t_1 t_2 t_3 t_4 t_5)^6 = 1$,
- (5) $\tau t_i = t_i \tau$ for $i = 1, \dots, 5$.

Let t_δ be a positive Dehn twist along the loop δ illustrated in Figure 2. Then $t_\delta = (t_1 t_2)^6$, this relation is called a chain relation.

2.2. Lantern Relation. Let us recall the definition of the lantern relation which will be used extensively in our construction of exotic 4-manifolds.

Let $\Sigma_{0,4}$ be a sphere with 4 boundary components.

FIGURE 2. Curves c_1 , c_2 , c_3 , c_4 , and c_5

Lemma 6. *If $\delta_1, \delta_2, \delta_3, \delta_4$ are the boundary curves of $\Sigma_{0,4}$ and α, β, γ are the simple closed curves as shown in Figure 3, then we have*

$$t_\gamma t_\beta t_\alpha = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4},$$

where t_{δ_i} , $1 \leq i \leq 4$, denote the Dehn twists about δ_i .

For a proof see ([15, 21]).

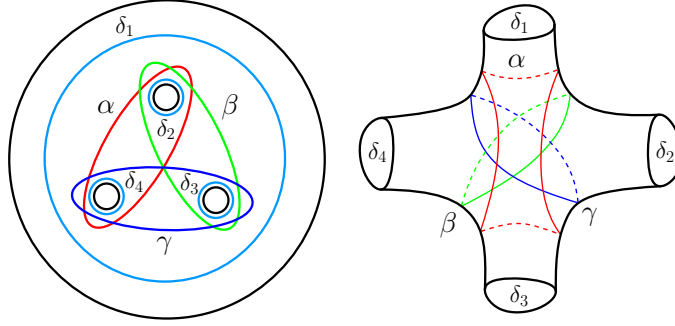


FIGURE 3. Curves defining lantern relation drawn two ways

The lantern relation in genus 2 surface implies $t_\gamma t_\beta t_\alpha = t_\beta t_\alpha t_\gamma = t_\alpha t_\gamma t_\beta$. This relation follows easily from the lantern relation plus the relation that each δ_i for $1 \leq i \leq 4$ commutes with each of $t_\gamma, t_\beta, t_\alpha$. Note that $t_\gamma t_\beta t_\alpha$ is not equal to $t_\beta t_\gamma t_\alpha$. We refer readers to the book of B. Farb and D. Margalit [15] for more details on mapping class group & lantern relation.

2.3. Lefschetz fibrations. In this section we recall the definition of Lefschetz fibrations over \mathbb{S}^2 and introduce three basic examples of complex genus two fibrations with no reducible fibers. We will also introduce Matsumoto's genus two Lefschetz fibrations over \mathbb{S}^2 with 8 singular fibers which are six irreducible fibers and two reducible fibers. They will later appear as the summands of the decomposable examples $X(n)$ for $2 \leq n \leq 6$.

Definition 7. Let X be a closed, oriented smooth 4-manifold. Lefschetz fibration of a smooth 4-manifold X comprises a smooth surjective map $f : X \rightarrow \mathbb{S}^2$, which is a submersion on the complement of finitely many points p_i in distinct fibers, at which there are local complex coordinates (compatible with fixed global orientations

on X and \mathbb{S}^2) with respect to which the map takes the form $(z_1, z_2) \mapsto z_1^2 + z_2^2$. We always assume that the fibers contain no (-1) -spheres (“relative minimality”) so in particular the fiber genus is always strictly positive.

By the hypotheses of good local complex models, each singular fiber of the Lefschetz fibration is a nodal curve with a unique nodal singularity, and it is obtained by shrinking a simple closed curve (the *vanishing cycle*) in the regular fiber to the nodal point of the singular fiber. They fall into two classes: irreducible fibers, where we collapse a non-separating cycle in the Riemann surface, and reducible fibers, where we collapse a separating cycle which gives the one-point union of smooth Riemann surfaces of smaller genera.

The existence of a Lefschetz fibration structure guarantees that X is a symplectic 4-manifold with an intrinsic symplectic form which takes the shape $\omega = \tau + Nf^*\omega_{\mathbb{S}}$ where τ is a closed form which is symplectic on the smooth fibres, and $\omega_{\mathbb{S}}$ is symplectic on the base $\mathbb{S}^2 \cong \mathbb{CP}^1$. The form is symplectic for sufficiently large N by the work of R. Gompf (cf. [19]). Topology of X is determined by a monodromy homomorphism $\psi_X : \pi_1(\mathbb{S}^2 \setminus \{f(p_i)\}) \rightarrow \Gamma_2$. The map ψ_X maps the generators of the fundamental group which encircle a single critical point once in an anticlockwise fashion to positive Dehn twists in the mapping class group. These Dehn twists are along the corresponding *vanishing cycles*. Thus the topology of X is completely encoded in an algebraic monodromy which is a word equal to the identity in the mapping class group, called a *positive relation*.

Let c_1, c_2, c_3, c_4 , and c_5 be the simple closed curves as in Figure 2. For convenience we shall denote the right handed Dehn twists t_{c_i} along the curve c_i by c_i . On the mapping class group Γ_2 , it is well known that the following positive relations hold,

$$(6) \quad \begin{aligned} (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 &= 1, \\ (c_1 c_2 c_3 c_4 c_5)^6 &= 1, \\ (c_1 c_2 c_3 c_4)^{10} &= 1. \end{aligned}$$

For each of the positive relations above, it follows that there exists the corresponding genus 2 Kähler Lefschetz fibrations over \mathbb{S}^2 with the total spaces $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2}$, $K3 \# 2\overline{\mathbb{CP}^2}$ and the Horikawa surface H respectively. (cf. [9, 33]).

2.4. Matsumoto’s genus two fibration. Matsumoto showed that $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}^2}$ has a genus 2 Lefschetz fibration with 6 irreducible singular fibers and 2 reducible singular fibers with a section of self-intersection -1 (cf. [25, 24]). The positive relation of the fibration is $(B_0 B_1 B_2 \delta)^2 = 1$, where B_0, B_1, B_2, δ are the curves indicated on Figure 4.

By using the classification of simple closed curves (cf. [15]), we know that there is only one nonseparating simple closed curve in surface S .

Proposition 8. *If α and β are any two nonseparating simple closed curves in a surface S , then there is a homeomorphism $\lambda : S \rightarrow S$ with $\lambda(\alpha) = \beta$.*

For a proof see [15].

This leads to the following useful proposition proven by the work of Akhmedov and Monden [2] which is by conjugating the global monodromy for Matsumoto’s genus 2 fibration by the well chosen mapping class of the Γ_2 which sends B_0 to

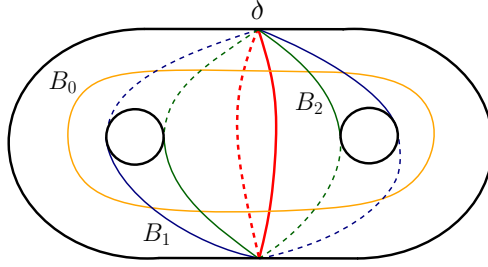


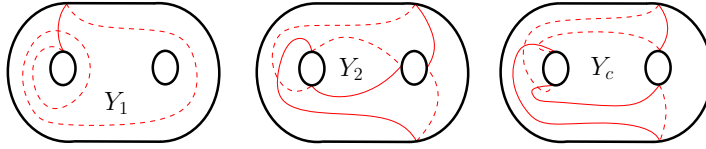
FIGURE 4. Curves for Matsumoto's genus 2 fibration

c_1 (both are nonseparating simple closed curves) we get a positive relation that contains $(c_1)^2$ which will later aid us in the construction of $X(7)$.

Proposition 9. *The Matsumoto's genus two Lefschetz fibration with the total space $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ can be given by a positive relation*

$$(7) \quad (c_1)^2 (Y_1 Y_2 Y_c)^2 = 1$$

which is acquired by conjugating global monodromy for Matsumoto's genus 2 fibration by the $\lambda = \iota\phi$ where $\phi = c_4^{-1}c_3^{-1}c_2^{-1}c_1^{-1}$ and ι is the vertical involution of the genus two surface with two fixed points.

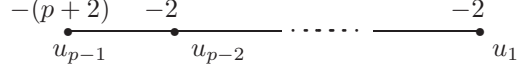
FIGURE 5. Special curves Y_1, Y_2, Y_c

For a detailed proof see [2].

2.5. Rational blowdown and Lantern relation substitution. Surgical procedure of rational blowdown was introduced by Fintushel and Stern in 1993 [16] and generalized to its present form by Jongil Park in 1997 [28] which allowed constructions of many important examples of exotic 4-manifolds due to its explicit interplay with the Seiberg-Witten invariants. Namely, if a closed smooth 4-manifold X contains a certain configuration C_p of transversally intersecting 2-spheres whose boundary is the lens space $L(p^2, 1 - p)$ then one can construct a new smooth 4-manifold X_p from X by replacing the interior of C_p with a rational ball B_p (as $L(p^2, 1 - p)$ bounds a rational ball B_p by Casson and Harer [8]) to construct a new manifold X_p . We say that X_p is obtained by rationally blowing down X along C_p . If one knows the Seiberg-Witten invariants of the original manifold X , then one can determine the Seiberg-Witten invariants of X_p .

Below we do the lightning review of the rational blowdown and refer the reader to [16] for detailed investigation.

Let $p \geq 2$ and C_p be the simply connected smooth 4-manifold obtained by plumbing the $(p - 1)$ disk bundles over the 2-sphere according to the following linear diagram:



where each node u_i of the linear diagram represents a disk bundle over 2-sphere with the given Euler number.

By the work of Casson and Harer [8], the boundary of C_p is the lens space $L(p^2, 1-p)$ which also bounds a rational ball B_p with $\pi_1(B_p) = \mathbb{Z}_p$ and $\pi_1(\partial B_p) \rightarrow \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold X_p is obtained by replacing C_p with B_p , i.e., $X_p = (X \setminus C_p) \cup B_p$. If X and $X \setminus C_p$ are simply connected, then so is X_p .

Note that $b_2^+(X_p) = b_2^+(X)$ so that rationally blowing down increases the signature while keeping b_2^+ . The following is easy to check.

Lemma 10. $b_2^+(X_p) = b_2^+(X)$, $\sigma(X_p) = \sigma(X) + (p-1)$, $c_1^2(X_p) = c_1^2(X) + (p-1)$, and $\chi_h(X_p) = \chi_h(X)$.

Proof. Notice that C_p is 4-manifold with negative definite intersection form, thus we have $b_2^+(X_p) = b_2^+(X)$ and $b_2^-(X_p) = b_2^-(X) - (p-1)$. Thus, $\sigma(X_p) = \sigma(X) + (p-1)$. Using the formulas $c_1^2 = 3\sigma + 2e$ and $\chi_h = (\sigma + e)/4$, we have $c_1^2(X_p) = 3\sigma(X_p) + 2e(X_p) = 3(\sigma(X) + (p-1)) + 2(e(X) - (p-1)) = c_1^2(X) + (p-1)$ and $\chi_h(X_p) = (\sigma(X) + (p-1) + e(X) - (p-1))/4 = \chi_h(X)$. \square

The following two theorems determines the effect of a rational blowdown on the Seiberg-Witten invariants.

Theorem 11. [16, 28]. *Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration C_p . If L is a SW basic class of X satisfying $L \cdot u_i = 0$ for any i with $1 \leq i \leq p-2$ and $L \cdot u_{p-1} = \pm p$, then L induces a SW basic class \bar{L} of X_p such that $SW_{X_p}(\bar{L}) = SW_X(L)$.*

Theorem 12. [16, 28] *If a simply connected smooth 4-manifold X contains a configuration C_p , then the SW-invariants of X_p are completely determined by those of X . That is, for any characteristic line bundle \bar{L} on X_p with $SW_{X_p}(\bar{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_p}(\bar{L})$.*

In our construction we will only use the rational blowdown surgery along configuration C_2 , i.e. the rational blowdowns along the -4 sphere.

The following theorem of H. Endo and Y. Gurtas in 2010 [13] connects the lantern relation substitution to the rational blowdown surgical operation defined above. Namely, one can perform the topological surgery of rational blowdown via algebraic monodromy substitution in the context of Lefschetz fibrations.

Theorem 13. *Let ϱ, ϱ' be positive relators of \mathcal{M}_g and $M_\varrho, M_{\varrho'}$ the corresponding Lefschetz fibrations over S^2 , respectively. If ϱ' is obtained by applying a lantern substitution to ϱ , then the 4-manifold $M_{\varrho'}$ is a rational blowdown of M_ϱ along a configuration $C_2 \subset M_\varrho$.*

Let us consider the following three cases of lantern substitution in Γ_2 .

- Making the lantern substitution $c_5 c_1^2 c_5$ for $c_3 \delta x$.

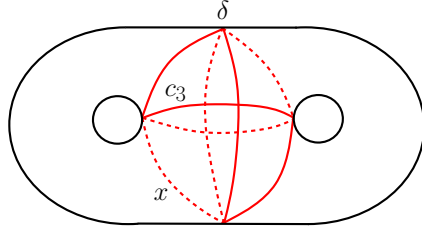
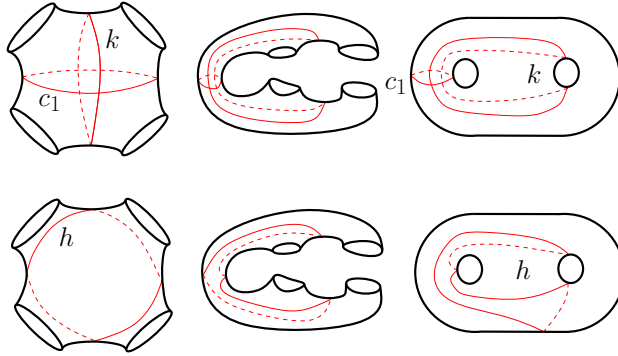
$$\begin{aligned}
& c_5 c_4 c_3 c_2 c_1 c_5 c_4 c_3 c_2 c_1 \\
& \sim c_5 (c_4) \cdot c_3 c_2 c_5 c_1 c_5 c_1 c_4 c_3 \cdot c_1^{-1} (c_2) \\
& \sim c_5 (c_4) \cdot c_3 c_2 \cdot c_5^2 c_1^2 \cdot c_4 c_3 \cdot c_1^{-1} (c_2) \\
& \xrightarrow{L} c_5 (c_4) \cdot c_3 c_2 \cdot c_3 \delta x \cdot c_4 c_3 \cdot c_1^{-1} (c_2)
\end{aligned}$$

- Making the lantern substitution $c_1 c_3 c_1 c_3$ for $\bar{k} \bar{h} c_5$.

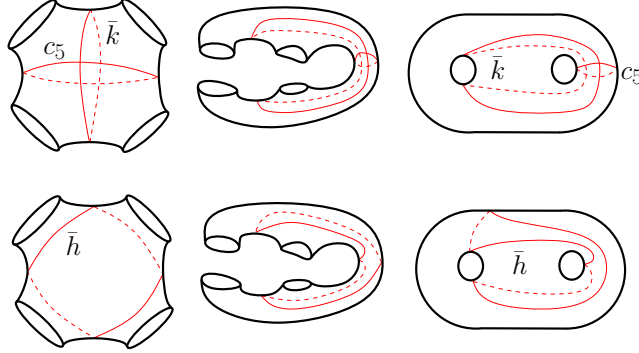
$$\begin{aligned}
& c_5 c_4 c_3 c_2 c_1 c_5 c_4 c_3 c_2 c_1 \\
& \sim c_5 c_4 c_5 c_3 c_2 c_4 c_1 c_3 c_2 c_1 \\
& \sim c_5 c_4 c_5 \cdot c_3 (c_2 c_4) \cdot c_1^2 c_3^2 \cdot c_1^{-1} (c_2) \\
& \xrightarrow{L} c_5 c_4 c_5 \cdot c_3 (c_2 c_4) \cdot \bar{k} \bar{h} c_5 \cdot c_1^{-1} (c_2)
\end{aligned}$$

- Making the lantern substitution $c_3 c_5^2 c_3$ for $c_1 k h$.

$$\begin{aligned}
& c_5 c_4 c_3 c_2 c_1 c_5 c_4 c_3 c_2 c_1 \\
& \sim c_5 (c_4) \cdot c_5 c_3 c_2 c_5 c_3 \cdot (c_4)_{c_3} \cdot c_1 c_2 c_1 \\
& \sim c_5 (c_4) \cdot c_3 (c_2) \cdot c_3^2 c_5^2 \cdot c_3^{-1} (c_4) \cdot c_1 c_2 c_1 \\
& \xrightarrow{L} c_5 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_3^{-1} (c_4) \cdot c_1 c_2 c_1
\end{aligned}$$

FIGURE 6. Special curves x, δ FIGURE 7. Special curves k, h

We will also need the following lemmas, which are due to R. Gompf, to analyze the symplectic 4-manifolds constructed in Section 3. For the proof we refer the reader to [18, 11].


 FIGURE 8. Special curves \bar{k} , \bar{h}

Lemma 14. *Let (X, V_X) be a relatively minimal smooth pair with V_X an embedded -4 sphere. If X contains a smoothly embedded exceptional sphere transversely intersecting the hypersurface V_X in a single positive point, then the manifold obtained under -4 blow-down of V_X is diffeomorphic to the blow-down of X along this sphere.*

Lemma 15. *Let (X, V_X) be a relatively minimal smooth pair with V_X an embedded -4 sphere. If X contain two disjoint smoothly embedded exceptional spheres each transversely intersecting the hypersurface V_X in a single positive point, then the manifold obtained under -4 blow-down of V_X is diffeomorphic to the blow-down of X along one of these spheres.*

2.6. Symplectic Minimality and Symplectic Kodaira Dimension. The notion of symplectic Kodaira dimension was introduced by D. McDuff and D. Salamon in 1996 [26] and discussed in detail by T.J. Li in [23]. We shall recall the definition of Kodaira dimension. In order to do so we first need to recall that a symplectic 4-manifold is called symplectically minimal if it does not contain any embedded symplectic spheres of square -1 . For a given symplectic 4-manifold (X, ω) , one can acquire its minimal model (X', ω') by blowing down a maximal disjoint collection of symplectic (-1) -spheres in X .

Definition 16. Let (X, ω) be a symplectic 4-manifold with minimal model (X', ω') and let $K_{X'} \in H^2(X'; \mathbb{Z})$ denote the canonical class of (X', ω') . Then the Kodaira dimension $\kappa^s(X, \omega)$ is

$$\kappa^s(X, \omega) = \begin{cases} -\infty & \text{if } K_{X'} \cdot [\omega'] < 0 \text{ or } K_{X'} \cdot K_{X'} < 0, \\ 0 & \text{if } K_{X'} \cdot [\omega'] = 0 \text{ and } K_{X'} \cdot K_{X'} = 0, \\ 1 & \text{if } K_{X'} \cdot [\omega'] > 0 \text{ and } K_{X'} \cdot K_{X'} = 0, \\ 2 & \text{if } K_{X'} \cdot [\omega'] > 0 \text{ and } K_{X'} \cdot K_{X'} > 0. \end{cases}$$

It was shown in [10] that the symplectic Kodaira dimension coincides with the complex Kodaira dimension when both are defined.

3. ANALYSIS OF GENUS 2 LEFSCHETZ FIBRATIONS

We now discuss the three different ways to describe the genus 2 Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2 . The first way is to obtain the $K3\#2\overline{\mathbb{CP}}^2$ fibration as

a double covering of $\mathbb{F}_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ branched along a smooth algebraic curve in the linear system $|6L|$, where L is a line in \mathbb{CP}^2 avoiding the blown-up point. This way of thinking about the $K3 \# 2\overline{\mathbb{CP}}^2$ fibration is discussed in detail in Lemma 6 of Akhmedov-Park [1]. Another way is to obtain $K3 \# 2\overline{\mathbb{CP}}^2$ fibration by holomorphically blowing up (the ordinary blow ups) twice the genus 2 pencil where the pencil itself is in turn acquired by the identity fiber summation of two copies of elliptic fibration on $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ along a regular torus fiber. Geometrically inclined readers will enjoy reading Proposition 7 of Akhmedov-Park [1] where this way of construction is given in detail. Finally, we portray here yet another way of thinking about $K3 \# 2\overline{\mathbb{CP}}^2$ over \mathbb{S}^2 . This new way is to obtain $K3 \# 2\overline{\mathbb{CP}}^2$ fibration by rationally blowing up twice the genus 2 Lefschetz fibration where the genus 2 Lefschetz fibration itself is in turn acquired by the identity fiber summation of genus 2 Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with genus 2 Lefschetz fibration on $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$.

Proposition 17. *The genus two Lefschetz fibration on $K3 \# 2\overline{\mathbb{CP}}^2$ over \mathbb{S}^2 can be acquired through performing two rational blowups on an untwisted fiber sum of genus two Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with the rational genus two Lefschetz fibration on $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$.*

Proof. Consider untwisted fiber sum (fiber sum with the identity map for the gluing diffeomorphism) of the Matsumoto's genus 2 Lefschetz fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ given by the positive relation $(B_0 B_1 B_2 \delta)^2 = 1$ in Section with the genus 2 rational Lefschetz fibration on $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ given by the positive relation $(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1$ in Section along a generic fiber Σ_2 . As the concatenation in the mapping class group corresponds to the symplectic fiber summing, the monodromy factorization of the resulting the Lefschetz fibration would be,

$$(8) \quad (B_0 B_1 B_2 \delta)^2 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1$$

Now, we will perform two rational blowups via lantern relation substitutions where we find $c_3 \delta x = \delta x c_3 = x c_3 \delta$ through elementary moves on the monodromy and substitute it with $c_1^2 c_5^2$.

Here and subsequently, when we perform monodromy computations, we denote the lantern relation substitution by \xrightarrow{L} , the braid relation substitution by \xrightarrow{B} , the conjugation by \xrightarrow{C} , and the arrangement using the commutativity by \sim respectively.

$$\begin{aligned} & (B_0 B_1 B_2 \delta)^2 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1 \\ & \sim B_0 B_1 B_2 \delta B_0 B_1 B_2 \delta \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \\ & \sim B_0 B_1 \cdot {}_{c_3^{-1}}(x) \cdot \delta B_0 B_1 \cdot {}_{c_3^{-1}}(x) \cdot \delta \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \\ & \{B_2 := {}_{c_3^{-1}}(x)\} \\ & \sim B_0 B_1 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \cdot {}_{c_3^{-1}}(x) \cdot \delta B_0 B_1 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \cdot {}_{c_3^{-1}}(x) \cdot \delta \\ & \{(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \text{ is central}\} \end{aligned}$$

$$\begin{aligned}
 & \sim B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot c_3 \cdot_{c_3^{-1}} (x) \cdot \delta B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot \\
 & c_3 \cdot_{c_3^{-1}} (x) \cdot \delta \\
 & \sim B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot x c_3 \delta \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot x c_3 \delta \\
 & \xrightarrow{L} B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot c_1^2 c_5^2 \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot x c_3 \delta \\
 & \xrightarrow{L} B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot c_1^2 c_5^2 \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot c_1^2 c_5^2 \\
 & \sim (B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot_{c_3} (c_2) \cdot c_1 \cdot c_1^2 c_5^2)^2 = \mathbf{X}(\mathbf{0})
 \end{aligned}$$

Topologically, the lantern relation substitution in the direction of finding $c_3 \delta x = \delta x c_3 = x c_3 \delta$ through elementary moves on the monodromy and substituting it with $c_1^2 c_5^2$ has the effect of rational blowup where one replaces the rational homology 4-ball with the tubular neighborhood of a (-4) -sphere. [13]

After two rational blowups, one arrives at the genus 2 Lefschetz fibration with the above monodromy $X(0)$ for the positive relation having 30 non-separating vanishing cycles which is transitive and has no separating vanishing cycles. (All singular fibers of genus 2 Lefschetz fibration $X(0)$ are irreducible)

By the Siebert and Tian's Theorem A in [34] on sufficient condition for holomorphicity of genus 2 Lefschetz fibrations over the \mathbb{S}^2 . We know that this genus 2 Lefschetz fibration is isomorphic to a holomorphic genus 2 Lefschetz fibration.

As Chakiris [9] assertion says every holomorphic fibrations of genus 2 without virtual reducible singular fibers is a fiber sum of three typical fibration (in our case either multiple of 20 or 30 irreducible singular fibers). This genus 2 holomorphic Lefschetz fibration with 30 irreducible singular fibers is clearly isomorphic to the fibration of $K3\#2\overline{\mathbb{CP}}^2$ with the above monodromy factorization. \square

Remark 18. By using the Theorem 3.5 in Auroux's [3] reformulation of the holomorphicity result obtained by Siebert and Tian in terms of the mapping class group factorizations indicates $X(0)$ is Hurwitz equivalent to a factorization of the form $(c_5 c_4 c_3 c_2 c_1)^6 = 1$ and thus the fibration is isomorphic to the one given in Akhmedov-Park's Lemma 6 and proposition 7 [1].

Now, we will provide propositions for the characterization of genus 2 Lefschetz fibrations with 20 irreducible singular fibers $(n, s) = (20, 0)$ and 18 irreducible singular fibers and 1 reducible singular fiber $(n, s) = (18, 1)$. Such characterizations of the genus 2 Lefschetz fibrations up to diffeomorphism will aid us in section 4 where we will consider all the possible decompositions of our decomposable exotic 4-manifolds examples. The proofs are adapted from Y. Sato's strategy which was effective in showing characterizations of seven and eight singular fibers genus 2 Lefschetz fibrations. (cf. [31])

Suppose that a genus 2 Lefschetz fibration $f : X \rightarrow \mathbb{S}^2$ has n irreducible singular fibers and s reducible singular fibers. Since the abelianization Γ_2^{ab} of the mapping class group Γ_2 is isomorphic to $\mathbb{Z}/10\mathbb{Z}$ (cf. [25]), we have $n + 2s \equiv 0 \pmod{10}$. As every singular fiber contributes 1 to the Euler characteristics $e(X)$, we have $e(X) = \# \text{singular fibers} + e(\mathbb{S}^2) e(\Sigma_2) = n + s - 4$. Moreover, for the signature

$\sigma(X)$, we have $\sigma(X) = -3n/5 - s/5$ by the Matsumoto's local signature formula [25].

Proposition 19 (Characterization of genus 2 Lefschetz fibration with 20 irreducible singular fibers). *Let $f : X \rightarrow \mathbb{S}^2$ has 20 irreducible singular fibers, then X is diffeomorphic to $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$.*

Proof. Let $f : X \rightarrow \mathbb{S}^2$ be a genus 2 Lefschetz fibration with n irreducible singular fibers and s reducible singular fibers. As the fibration we are interested in has $(20, 0)$ for (n, s) pair, its Euler characteristic and signature numbers are equal to $e(X) = 16$ and $\sigma(X) = -12$ with $c_1^2(X) = 2e(X) + 3\sigma(X) = -4$. Next, we will determine (b_2^+, b_2^-, b_1) for X . Since $2 - 2b_1 + 2b_2^+ = e + \sigma = 4$ we obtain $b_2^+ = b_1 + 1$. Let H be the subspace of $H_1(\Sigma_2; \mathbb{R})$ generated by the vanishing cycles of X . Here, Σ_2 denotes the reference fiber of genus 2. Since a Lefschetz fibration over \mathbb{S}^2 must have a nonseparating vanishing cycle [35], we have $\dim H \geq 1$. And since $H_1(X; \mathbb{R}) = H_1(\Sigma_2; \mathbb{R})/H$, we acquire that $b_1(X) = 4 - \dim H \leq 3$. Thus, we have that $1 \leq b_2^+ = b_1 + 1 \leq 4$, and therefore gives four possible triple for $(b_2^+, b_2^-, b_1) = (1, 13, 0), (2, 14, 1), (3, 15, 2)$ or $(4, 16, 3)$. Suppose that $b_2^+ > 1$. We will show this is impossible as $K_X^2 = c_1^2 = 3\sigma + 2e = -4$. Hence it follows from Theorem 0.2 in [39] that X is not minimal, that is, $f : X \rightarrow \mathbb{S}^2$ is a non-minimal genus 2 Lefschetz fibration with $(n, s) = (20, 0)$. However, by the Table 1, of the geography of non-minimal genus 2 Lefschetz fibrations over \mathbb{S}^2 [30], there is not any $b_2^+ > 1$ non-minimal genus 2 Lefschetz fibration over \mathbb{S}^2 with $(n, s) = (20, 0)$. Therefore, a genus 2 Lefschetz fibration $f : X \rightarrow \mathbb{S}^2$ with $(n, s) = (20, 0)$ satisfies $(b_2^+, b_2^-, b_1) = (1, 13, 0)$.

Next we will show X is a rational surface. Suppose that X is not a rational surface. Let \tilde{X} be the minimal model of X . Since $b_2^+(\tilde{X}) = 1$ and $b_1(\tilde{X}) = 0$, we have that $c_1^2(\tilde{X}) = 3\sigma(\tilde{X}) + 2e(\tilde{X}) = 5b_2^+(\tilde{X}) - b_2^-(\tilde{X}) - 4b_1(\tilde{X}) + 4 = 9 - b_2^-(\tilde{X})$. Moreover, since \tilde{X} is a minimal symplectic 4-manifold with $b_2^+ = 1$ and \tilde{X} is not rational nor ruled, it follows from [22] that \tilde{X} satisfies $c_1^2(\tilde{X}) \geq 0$. Hence, we have $b_2^-(\tilde{X}) \leq 9$. Since X is not rational nor ruled and X admits a genus 2 Lefschetz fibration over \mathbb{S}^2 , it follows from Theorem 3.1 [30] that X contains at most two 2-spheres with self-intersection number -1 essentially. Therefore, we have that $b_2^-(X) \leq 11$. This is in contradiction with $b_2^-(X) = 13$. Thus, X is a rational surface, and X is diffeomorphic to $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. \square

Proposition 20 (Characterization of genus 2 Lefschetz fibration with 18 irreducible singular fibers and 1 reducible singular fiber). *Let $f : X \rightarrow \mathbb{S}^2$ has 18 irreducible singular fibers and one reducible singular fiber, then X is diffeomorphic to $\mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$.*

Proof. Let $f : X \rightarrow \mathbb{S}^2$ be a genus 2 Lefschetz fibration with n irreducible singular fibers and s reducible singular fibers. As the fibration we are interested in has $(18, 1)$ for (n, s) pair, its Euler characteristic and signature numbers are equal to $e(X) = 15$ and $\sigma(X) = -11$ with $c_1^2(X) = 2e(X) + 3\sigma(X) = -3$. We note that X is non-spin as there is a reducible fiber (i.e. $s = 1$) [36]. Next, we will determine (b_2^+, b_2^-, b_1) for X . Since $2 - 2b_1 + 2b_2^+ = e + \sigma = 4$ we obtain $b_2^+ = b_1 + 1$. Let H be the subspace of $H_1(\Sigma_2; \mathbb{R})$ generated by the vanishing cycles of X . Here, Σ_2 denotes the reference fiber of genus 2. Since a Lefschetz fibration over \mathbb{S}^2 must have a nonseparating vanishing cycle [35], we have $\dim H \geq 1$. And

since $H_1(X; \mathbb{R}) = H_1(\Sigma_2; \mathbb{R})/H$, we acquire that $b_1(X) = 4 - \dim H \leq 3$. Thus, we have that $1 \leq b_2^+ = b_1 + 1 \leq 4$, and therefore gives four possible triple for $(b_2^+, b_2^-, b_1) = (1, 12, 0), (2, 13, 1), (3, 14, 2)$ or $(4, 15, 3)$. Suppose that $b_2^+ > 1$. We will show this is impossible as $K_X^2 = c_1^2 = 3\sigma + 2e = -3$. Hence it follows from Theorem 0.2 in [39] that X is not minimal, that is, $f : X \rightarrow \mathbb{S}^2$ is a non-minimal genus 2 Lefschetz fibration with $(n, s) = (18, 1)$. However, by the Table 1, of the geography of non-minimal genus 2 Lefschetz fibrations over \mathbb{S}^2 [30], there is not any $b_2^+ > 1$ non-minimal genus 2 Lefschetz fibration over \mathbb{S}^2 with $(n, s) = (18, 1)$. Therefore, a genus 2 Lefschetz fibration $f : X \rightarrow \mathbb{S}^2$ with $(n, s) = (18, 1)$ satisfies $(b_2^+, b_2^-, b_1) = (1, 12, 0)$.

Next we will show X is a rational surface. Suppose that X is not a rational surface. Let \tilde{X} be the minimal model of X . Since $b_2^+(\tilde{X}) = 1$ and $b_1(\tilde{X}) = 0$, we have that $c_1^2(\tilde{X}) = 3\sigma(\tilde{X}) + 2e(\tilde{X}) = 5b_2^+(\tilde{X}) - b_2^-(\tilde{X}) - 4b_1(\tilde{X}) + 4 = 9 - b_2^-(\tilde{X})$. Moreover, since \tilde{X} is a minimal symplectic 4-manifold with $b_2^+ = 1$ and \tilde{X} is not rational nor ruled, it follows from [22] that \tilde{X} satisfies $c_1^2(\tilde{X}) \geq 0$. Hence, we have $b_2^-(\tilde{X}) \leq 9$. Since X is not rational nor ruled and X admits a genus 2 Lefschetz fibration over \mathbb{S}^2 , it follows from Theorem 3.1 [30] that X contains at most two 2-spheres with self-intersection number -1 essentially. Therefore, we have that $b_2^-(X) \leq 11$. This is in contradiction with $b_2^-(X) = 12$. Thus, X is a rational surface, and X is diffeomorphic to $\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$. \square

4. CONSTRUCTION OF DECOMPOSABLE EXOTIC 4-MANIFOLDS

In this section, we construct simply-connected, minimal symplectic 4-manifolds $X(n)$ for $2 \leq n \leq 6$ homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (21 - n)\overline{\mathbb{CP}^2}$ by starting from $K3 \# 2\overline{\mathbb{CP}^2}$ and applying a sequence of six rational blowdowns via lantern relation substitutions. These $X(n)$ are constructed in terms of methodology similar to Akhmedov-Park examples in [1] and thus shares geometric properties but we will use different monodromy and Hurwitz moves which will help us to show the decomposability of $X(n)$.

Theorem 21 (Construction of $X(n)$ for $2 \leq n \leq 6$). *Let $X(0)$ denote the total space of the genus two Lefschetz fibration on $K3 \# 2\overline{\mathbb{CP}^2}$ over S^2 given by the positive relation $(B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_1^2 c_5^2)^2 = 1$ in M_2 . There exist irreducible, simply-connected, symplectic 4-manifolds $X(n)$ for $2 \leq n \leq 6$ which are homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (21 - n)\overline{\mathbb{CP}^2}$ for $2 \leq n \leq 6$ that can be obtained by applying six lantern substitutions to the global monodromy relation of $X(0)$. Moreover, $X(n)$ for $2 \leq n \leq 6$ are minimal symplectic 4-manifolds with the symplectic Kodaira dimension $\kappa^s(X(2)) = 1$ and $\kappa^s(X(n)) = 2$ for $3 \leq n \leq 6$.*

To prove this theorem, we need to prove the following two lemmas first.

Lemma 22 (Monodromy of $Z(m)$ for $1 \leq m \leq 4$). *The global monodromy of genus 2 Lefschetz fibration on $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2}$ over S^2 given by the relation $Z(0) = (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1$ can be braid substituted to contain four lantern relations.*

Proof. We start with the identity word: $(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1$

$$Z(0) = (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = 1$$

$$\begin{aligned}
& \sim c_1 c_2 c_3 c_4 c_5 c_5 c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4 c_5 c_5 c_4 c_3 c_2 c_1 \\
& \sim c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot c_3(c_2) \cdot c_1^2 c_3^2 \cdot c_3^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 c_2 c_1 \\
& \xrightarrow{L} c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot c_3(c_2) \cdot c_5 \bar{k} \bar{h} \cdot c_3^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 c_2 c_1 = \mathbf{Z(1)} \\
& \sim c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot c_1 \cdot c_3 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1}(\bar{k} \bar{h}) \cdot c_3^{-1} c_1^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot c_5^2 c_1^2 \cdot c_4 \cdot c_3 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1}(\bar{k} \bar{h}) \cdot c_3^{-1} c_1^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) \\
& \xrightarrow{L} c_1(c_2) \cdot c_3 c_4 \cdot \delta x c_3 \cdot c_4 \cdot c_3 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1}(\bar{k} \bar{h}) \cdot c_3^{-1} c_1^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) = \mathbf{Z(2)} \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_3 \cdot c_3^{-1} c_1^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_1^{-1}(c_2) \cdot c_3 \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_1^{-1}(c_2) \cdot c_3(c_4) \cdot c_3^2 c_5^2 \cdot c_3^{-1}(c_4) \cdot c_1^{-1}(c_2) \\
& \xrightarrow{L} c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_1^{-1}(c_2) \cdot c_3(c_4) \cdot k h c_1 \cdot c_3^{-1}(c_4) \cdot c_1^{-1}(c_2) = \mathbf{Z(3)} \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_1^{-1}(c_2) \cdot c_3(c_4) \cdot k h \cdot c_3^{-1}(c_4) \cdot c_2 c_1 \\
& \sim c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_1^{-1}(c_2) \cdot c_2 \cdot c_2^{-1} c_3(c_4) \cdot c_2^{-1}(k h) \cdot c_2^{-1} c_3^{-1}(c_4) \cdot c_1 \\
& \xrightarrow{B} c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 c_1 \cdot c_2^{-1} c_3(c_4) \cdot c_2^{-1}(k h) \cdot c_2^{-1} c_3^{-1}(c_4) \cdot c_1 \\
& \sim c_1(c_2) \cdot c_3 \cdot c_4(\delta x) \cdot c_4 \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 c_1 \cdot c_2^{-1} c_3(c_4) \cdot c_2^{-1}(k h) \cdot c_2^{-1} c_3^{-1}(c_4) \cdot c_1 \\
& \xrightarrow{B} c_1(c_2) \cdot c_3 \cdot c_4(\delta x) \cdot c_3 c_4 \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 c_1 \cdot c_2^{-1} c_3(c_4) \cdot c_2^{-1}(k h) \cdot c_2^{-1} c_3^{-1}(c_4) \cdot c_1 \\
& \sim c_1(c_2) \cdot c_3 \cdot c_4(\delta x) \cdot c_3 c_4 \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 \cdot c_1^2 \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(k h) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \\
& \sim c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2 \cdot c_4 \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 \cdot c_1^2 \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(k h) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \\
& \sim c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_3^2 \cdot c_5 \cdot c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_2 \cdot c_1^2 \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(k h) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \\
& \sim c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k} \bar{h}) \cdot c_3^2(c_2) \cdot c_1^2 c_3^2 \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(k h) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4)
\end{aligned}$$

$$\xrightarrow{L} c_1(c_2) \cdot c_3 c_4 (\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_3^2(c_2) \cdot c_5 \bar{k}\bar{h} \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) = \mathbf{Z(4)}$$

□

We collect positive relations of $Z(m)$ for $0 \leq m \leq 4$ below,

- $(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = \mathbf{Z(0)}$
- $c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot c_3(c_2) \cdot c_5 \bar{k}\bar{h} \cdot c_3^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 c_2 c_1 = \mathbf{Z(1)}$
- $c_1(c_2) \cdot c_3 c_4 \cdot \delta x c_3 \cdot c_4 \cdot c_3 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1}(\bar{k}\bar{h}) \cdot c_3^{-1} c_1^{-1}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot c_1^{-1}(c_2) = \mathbf{Z(2)}$
- $c_1(c_2) \cdot c_3 c_4 \cdot \delta x \cdot c_3(c_4) \cdot c_3^2 c_1^{-1}(c_2) \cdot c_5 \cdot c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_1^{-1}(c_2) \cdot c_3(c_4) \cdot kh c_1 \cdot c_3^{-1}(c_4) \cdot c_1^{-1}(c_2) = \mathbf{Z(3)}$
- $c_1(c_2) \cdot c_3 c_4 (\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_3^2(c_2) \cdot c_5 \bar{k}\bar{h} \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) = \mathbf{Z(4)}$

We see that $Z(m)$ for $1 \leq m \leq 4$ are rational blowdown copies of $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2}$ by the Theorem 3.1 in [13] constructed in the same methodology as in the Endo-Gurtas and thus homeomorphic to $\mathbb{CP}^2 \# (13 - m)\overline{\mathbb{CP}^2}$ for $0 \leq m \leq 4$ (cf. [13]) except that we have avoided in using the conjugation move \xrightarrow{C} which will facilitate fiber sum splitting of $X(n)$ for $2 \leq n \leq 6$ constructed below. It is also worth noting that $Z(1)$ is diffeomorphic to $\mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$ by Proposition 20. After this we can characterize the $Z(2), Z(3), Z(4)$ only up to homeomorphism types of $\mathbb{CP}^2 \# (12 - m)\overline{\mathbb{CP}^2}$ for $2 \leq m \leq 4$.

Combining the above monodromy computations for $X(0)$ and $Z(m)$. We can now find six lantern relations on $K3 \# 2\overline{\mathbb{CP}^2}$ which allows the fiber sum decomposability.

Lemma 23 (Monodromy of $X(n)$ for $2 \leq n \leq 6$). *The global monodromy of genus 2 Lefschetz fibration on $K3 \# 2\overline{\mathbb{CP}^2}$ over S^2 given by the relation $X(0)$ in Proposition 17 can be conjugated and braid substituted to contain six lantern relations.*

Proof. We begin with $\mathbf{X(0)} = (B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_1^2 c_5^2)^2 = 1$

$$\sim B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_1^2 c_5^2 \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_1^2 c_5^2$$

$$\xrightarrow{L} B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_1^2 c_5^2 \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot x c_3 \delta = \mathbf{X(1)}$$

$$\xrightarrow{L} B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot x c_3 \delta \cdot B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot x c_3 \delta = \mathbf{X(2)}$$

$$\sim B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_3 \cdot c_3^{-1}(x) \cdot \delta B_0 B_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 \cdot c_3(c_2) \cdot c_1 \cdot c_3 \cdot c_3^{-1}(x) \cdot \delta$$

$$\sim B_0 B_1 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \cdot c_3^{-1}(x) \cdot \delta B_0 B_1 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \cdot c_3^{-1}(x) \cdot \delta$$

$$\sim B_0 B_1 \cdot_{c_3^{-1}}(x) \cdot \delta B_0 B_1 \cdot_{c_3^{-1}}(x) \cdot \delta \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \\ \{(c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1) \text{ is central}\}$$

$$\sim B_0 B_1 B_2 \delta B_0 B_1 B_2 \delta \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \cdot c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1 \\ \{B_2 :=_{c_3^{-1}}(x)\}$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = \mathbf{X(2)}$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot c_1 c_2 c_3 c_4 c_5 c_5 c_4 c_3 c_2 c_1 c_1 c_2 c_3 c_4 c_5 c_5 c_4 c_3 c_2 c_1$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot_{c_3}(c_2) \cdot c_1^2 c_3^2 \cdot_{c_3^{-1}}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 c_2 c_1$$

$$\xrightarrow{L} (B_0 B_1 B_2 \delta)^2 \cdot c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot_{c_3}(c_2) \cdot c_5 \bar{k} \bar{h} \cdot_{c_3^{-1}}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 c_2 c_1 = \mathbf{X(3)}$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot c_1 \cdot_{c_3 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1}}(\bar{k} \bar{h}) \cdot_{c_3^{-1} c_1^{-1}}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot_{c_1^{-1}}(c_2)$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot c_5^2 c_1^2 \cdot c_4 \cdot_{c_3 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1}}(\bar{k} \bar{h}) \cdot_{c_3^{-1} c_1^{-1}}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot_{c_1^{-1}}(c_2)$$

$$\xrightarrow{L} (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x c_3 \cdot c_4 \cdot_{c_3 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1}}(\bar{k} \bar{h}) \cdot_{c_3^{-1} c_1^{-1}}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot_{c_1^{-1}}(c_2) = \mathbf{X(4)}$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_3 \cdot_{c_3^{-1} c_1^{-1}}(c_2)}(c_2) \cdot c_4 c_5 c_5 c_4 c_3 \cdot_{c_1^{-1}}(c_2)$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_1^{-1}}(c_2) \cdot_{c_3}(c_4) \cdot c_4 c_5 c_5 c_4 c_3 \cdot_{c_1^{-1}}(c_2)$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_1^{-1}}(c_2) \cdot_{c_3}(c_4) \cdot c_3^2 c_5^2 \cdot_{c_3^{-1}}(c_4) \cdot_{c_1^{-1}}(c_2)$$

$$\xrightarrow{L} (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_1^{-1}}(c_2) \cdot_{c_3}(c_4) \cdot k h c_1 \cdot_{c_3^{-1}}(c_4) \cdot_{c_1^{-1}}(c_2) = \mathbf{X(5)}$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_1^{-1}}(c_2) \cdot_{c_3}(c_4) \cdot k h \cdot_{c_3^{-1}}(c_4) \cdot c_2 c_1$$

$$\sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot_{c_1^{-1}}(c_2) \cdot c_2 \cdot_{c_2^{-1} c_3}(c_4) \cdot_{c_2^{-1}}(k h) \cdot_{c_2^{-1} c_3^{-1}}(c_4) \cdot c_1$$

$$\xrightarrow{B} (B_0 B_1 B_2 \delta)^2 \cdot_{c_1}(c_2) \cdot c_3 c_4 \cdot \delta x \cdot_{c_3}(c_4) \cdot_{c_3^2 c_1^{-1}}(c_2) \cdot c_5 \cdot_{c_1^{-1} c_3}(\bar{k} \bar{h}) \cdot c_2 c_1 \cdot_{c_2^{-1} c_3}(c_4) \cdot_{c_2^{-1}}(k h) \cdot_{c_2^{-1} c_3^{-1}}(c_4) \cdot c_1$$

$$\begin{aligned}
 & \sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 \cdot c_4} (\delta x) \cdot_{c_4 \cdot c_3} (c_4) \cdot_{c_3^2 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_2 c_1 \cdot c_2^{-1} c_3} (c_4) \cdot \\
 & c_2^{-1} (kh) \cdot_{c_2^{-1} c_3^{-1}} (c_4) \cdot c_1 \\
 & \xrightarrow{B} (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 \cdot c_4} (\delta x) \cdot_{c_3 c_4} (c_3 c_4) \cdot_{c_3^2 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_2 c_1 \cdot c_2^{-1} c_3} (c_4) \cdot \\
 & c_2^{-1} (kh) \cdot_{c_2^{-1} c_3^{-1}} (c_4) \cdot c_1 \\
 & \sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 \cdot c_4} (\delta x) \cdot_{c_3 c_4} (c_3 c_4) \cdot_{c_3^2 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_2 \cdot c_1^2 \cdot c_1^{-1} c_2^{-1} c_3} (c_4) \cdot \\
 & c_1^{-1} c_2^{-1} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) \\
 & \sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3^2 \cdot c_4 \cdot c_3^2 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_2 \cdot c_1^2 \cdot c_1^{-1} c_2^{-1} c_3} (c_4) \cdot \\
 & c_1^{-1} c_2^{-1} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) \\
 & \sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3^2 (c_4) \cdot c_3^4 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_2 \cdot c_1^2} \cdot c_1^{-1} c_3^{-1} (c_4) \cdot \\
 & c_1^{-1} c_2^{-1} c_3^{-1} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) \\
 & \sim (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3^2 (c_4) \cdot c_3^4 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_3^2 c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_3^2 (c_2) \cdot c_1^2 c_3^2} \cdot \\
 & c_1^{-1} c_2^{-1} c_3^{-1} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) \\
 & \xrightarrow{L} (B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3^2 (c_4) \cdot c_3^4 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_3^2 c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_3^2 (c_2) \cdot c_5 \bar{k} \bar{h}} \cdot \\
 & c_1^{-1} c_2^{-1} c_3^{-1} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) = \mathbf{X(6)}
 \end{aligned}$$

□

We collect positive relations of $X(n)$ for $2 \leq n \leq 6$ below,

- $(B_0 B_1 B_2 \delta)^2 \cdot (c_1 c_2 c_3 c_4 c_5^2 c_4 c_3 c_2 c_1)^2 = \mathbf{X(2)}$
- $(B_0 B_1 B_2 \delta)^2 \cdot c_1 c_2 c_3 c_4 c_5 c_5 c_4 \cdot_{c_3} (c_2) \cdot_{c_5 \bar{k} \bar{h}} \cdot_{c_3^{-1}} (c_2) \cdot_{c_4 c_5 c_5 c_4 c_3 c_2 c_1} = \mathbf{X(3)}$
- $(B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3} (c_4) \cdot_{c_3 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1}} (\bar{k} \bar{h}) \cdot_{c_3^{-1} c_1^{-1}} (c_2) \cdot_{c_4 c_5 c_5 c_4 c_3} \cdot_{c_1^{-1}} (c_2) = \mathbf{X(4)}$
- $(B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3} (c_4) \cdot_{c_3^2 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_1^{-1}} (c_2) \cdot_{c_3} (c_4) \cdot_{kh c_1 \cdot c_3^{-1}} (c_4) \cdot_{c_1^{-1}} (c_2) = \mathbf{X(5)}$
- $(B_0 B_1 B_2 \delta)^2 \cdot_{c_1} (c_2) \cdot_{c_3 c_4} (\delta x) \cdot_{c_3^2 (c_4) \cdot c_3^4 c_1^{-1}} (c_2) \cdot_{c_5 \cdot c_3^2 c_1^{-1} c_3} (\bar{k} \bar{h}) \cdot_{c_3^2 (c_2) \cdot c_5 \bar{k} \bar{h}} \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (kh) \cdot_{c_1^{-1} c_2^{-1} c_3^{-1}} (c_4) = \mathbf{X(6)}$

We now give a proof of the main theorem of this section Theorem 21.

Proof. Let $X(n)$ for $2 \leq n \leq 6$ be the symplectic 4-manifold obtained from $K3\#2\overline{\mathbb{CP}}^2$ by applying a sequence of six lantern relation substitutions as in Lemma 23 above.

We first compute the topological invariants to determine the homeomorphism types of $X(n)$ for $2 \leq n \leq 6$.

$$\begin{aligned}
e(X(n)) &= \# \text{singular fibers} + e(\mathbb{S}^2)e(\Sigma_2) = (30 - n) + 2(-2) = 26 - n, \\
\sigma(X(n)) &= -\frac{3}{5}n - \frac{1}{5}s = -\frac{3}{5}(30 - 2n) - \frac{1}{5}(n) = -18 + n, \\
c_1^2(X(n)) &:= 3\sigma(X(n)) + 2e(X(n)) = n - 2, \\
\chi(X(n)) &:= (e(X(n)) + \sigma(X(n)))/4 = 2
\end{aligned}$$

$X(n)$ for $2 \leq n \leq 6$ are simply-connected as $X(0) = K3\#2\overline{\mathbb{CP}}^2$ and $X(n-1) \setminus C_2$ are simply-connected. They have the Euler characteristic $e(X(n)) = e(K3\#2\overline{\mathbb{CP}}^2) - n = (26) - n$ with the signature $\sigma(X(n)) = \sigma(K3\#2\overline{\mathbb{CP}}^2) + n = (-18) + n$. Note that they all are non-spin as there are reducible fibers [36]. All together, $X(n)$ for $2 \leq n \leq 6$ are homeomorphic to $3\mathbb{CP}^2\#(21-n)\overline{\mathbb{CP}}^2$ from Freedman's classification theorem (cf. [14]).

Using the blow up formula for the Seiberg-Witten function [17], we have $SW_{K3\#2\overline{\mathbb{CP}}^2} = SW_{K3} \cdot \prod_{i=1}^2 (e^{E_i} + e^{-E_i}) = (e^{E_1} + e^{-E_1})(e^{E_2} + e^{-E_2})$, where E_i is an exceptional class coming from the i^{th} blow up. Thus the set of basic classes of $K3\#2\overline{\mathbb{CP}}^2$ are given by $\pm E_1 \pm E_2$, and the Seiberg-Witten invariants on these classes are ± 1 . After performing one rational blowdown along a copy of the configuration C_2 , the resulting manifold is diffeomorphic to $K3\#\overline{\mathbb{CP}}^2$ by Lemma 15. Thus, the only basic classes are $\pm E$, where $E \in H^2(K3\#\overline{\mathbb{CP}}^2)$ is the poincaré dual of the homology class of the exceptional sphere, which descends from the top classes $\pm(E_1 + E_2)$ in $K3\#2\overline{\mathbb{CP}}^2$. Next, using the Corollary 8.6 in [16], we see that X has Seiberg-Witten simple type. By applying Theorem 11 and Theorem 12, we completely determine the Seiberg-Witten invariants of X using the basic classes and invariants of $K3\#\overline{\mathbb{CP}}^2$: Up to sign the symplectic manifold X has only one basic class which descends from the canonical class of $K3\#\overline{\mathbb{CP}}^2$. By Theorem 12 (or by Taubes theorem [38]), the value of the Seiberg-Witten function on these classes, $\pm K_{X(n)}$, are ± 1 .

By using Fintushel-Stern's rational blowdown formula we can also determine the Seiberg-Witten invariants of $X(n)$ for $2 \leq n \leq 6$ directly by computing the algebraic intersection number of the classes $\pm E_1 \pm E_2$ with the classes of -4 spheres of six C_2 configurations. Note that these -4 spheres are the components of the singular fibers of $K3\#2\overline{\mathbb{CP}}^2$. As three regions on the genus two surface, where the rational blowdowns are performed always intersect the two exceptional divisors once (cf. [1]), we compute the intersection numbers as follows: Let S denote the homology class of -4 sphere of C_2 . We have $S \cdot E_1 = S \cdot E_2 = 1$. Consequently, $S \cdot \pm(E_1 + E_2) = \pm 2$ and $S \cdot \pm(E_1 - E_2) = 0$. Since among the four classes $\pm E_1 \pm E_2$ only $E_1 + E_2$ and $-(E_1 + E_2)$ have intersection ± 2 with -4 spheres of C_2 , it follows from Theorem 11 that these are only two classes that descend to $X(n)$ for $2 \leq n \leq 6$.

Next, we apply the connected sum theorem for the Seiberg-Witten invariant and show that SW function is trivial for $3\mathbb{CP}^2\#(21-n)\overline{\mathbb{CP}}^2$ for $2 \leq n \leq 6$. Since the Seiberg-Witten invariants are diffeomorphism invariants, we conclude that $X(n)$ for $2 \leq n \leq 6$ are not diffeomorphic to $3\mathbb{CP}^2\#(21-n)\overline{\mathbb{CP}}^2$ for $2 \leq n \leq 6$.

Using the Seiberg-Witten basic classes, the minimality of $X(n)$ for $2 \leq n \leq 6$ follows from the fact that $X(n)$ for $2 \leq n \leq 6$ has no two basic classes K and K'

such that $(K - K')^2 = -4$. Notice that $(K_{X(n)} - (-K_{X(n)}))^2 = 4(K_X(n))^2 = 16$ for $2 \leq n \leq 6$ in our case.

The symplectic Kodaira dimension $\kappa^s(X(n))$ for $2 \leq n \leq 6$ are equal to $\kappa^s = 1$ for $n = 2$ and $\kappa^s = 2$ for $3 \leq n \leq 6$. The $X(2)$ has $\kappa^s(X(2)) = 1$ since it is a minimal exotic copy of $3\mathbb{CP}^2 \# 19\overline{\mathbb{CP}^2}$ (cf. [10, 23]). Finally, $\kappa^s(X(n)) = 2$ for $3 \leq n \leq 6$ since they are also minimal and have $c_1^2(X(n)) \geq 0$.

Thus the $X(n)$ for $1 \leq n \leq 6$ are simply-connected, symplectic 4-manifolds homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# (21 - n)\overline{\mathbb{CP}^2}$ with $b_2^+ = 3$ and symplectically minimal for $2 \leq n \leq 6$ with symplectic Kodaira dimension $\kappa^s = 0$ for $n = 1$, $\kappa^s = 1$ for $n = 2$ and $\kappa^s = 2$ for $3 \leq n \leq 6$. \square

5. CONSTRUCTION OF $X(7)$

In this section, we will find one more lantern relation from positive relation of $X(6)$ by replacing the Matsumoto's fibration summand $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}^2}$ with its globally conjugated copy having the positive relation of

$$(9) \quad (c_1)^2(Y_1 Y_2 Y_c)^2 = 1$$

which is introduced in Proposition 9.

With this relation we can replace the Matsumoto's fibration summand word $(B_0 B_1 B_2 \delta)^2$ in $X(6)$ with $(c_1)^2(Y_1 Y_2 Y_c)^2$ which would allow us to perform the seventh lantern substitution.

We can now perform one more lantern substitution via the following Hurwitz move.

Proof. We begin with relation of $X(6)$

$$\begin{aligned} \mathbf{X}(6) &= (B_0 B_1 B_2 \delta)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_3^2(c_2) \cdot \\ &c_5 \bar{k}\bar{h} \cdot c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) = 1 \\ &\sim (c_1)^2(Y_1 Y_2 Y_c)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_3^2(c_2) \cdot c_5 \bar{k}\bar{h} \cdot \\ &c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \\ &\xrightarrow{C} (Y_1 Y_2 Y_c)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 \cdot c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_3^2(c_2) \cdot c_5 \bar{k}\bar{h} \cdot \\ &c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \cdot c_1^2 \\ &\sim (Y_1 Y_2 Y_c)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_5 c_3^2(c_2) \cdot c_5^2 \bar{k}\bar{h} \cdot \\ &c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_1^{-1} c_2^{-1}(kh) \cdot c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \cdot c_1^2 \\ &\sim (Y_1 Y_2 Y_c)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_5 c_3^2(c_2) \cdot c_5^2(\bar{k}\bar{h}) \cdot \\ &c_3^2 c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_5^2 c_1^{-1} c_2^{-1}(kh) \cdot c_5^2 c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \cdot c_5^2 c_1^2 \\ &\xrightarrow{L} (Y_1 Y_2 Y_c)^2 \cdot c_1(c_2) \cdot c_3 c_4(\delta x) \cdot c_3^2(c_4) \cdot c_3^4 c_1^{-1}(c_2) \cdot c_5 c_3^2 c_1^{-1} c_3(\bar{k}\bar{h}) \cdot c_5 c_3^2(c_2) \cdot c_5^2(\bar{k}\bar{h}) \cdot \\ &c_3^2 c_1^{-1} c_2^{-1} c_3(c_4) \cdot c_5^2 c_1^{-1} c_2^{-1}(kh) \cdot c_5^2 c_1^{-1} c_2^{-1} c_3^{-1}(c_4) \cdot c_3 \delta x = \mathbf{X}(7) \end{aligned}$$

\square

We note that the total space of the $X(7)$ is a symplectically minimal exotic copy of $3\mathbb{CP}^2 \# 14\overline{\mathbb{CP}}^2$ with $b_2^+ = 3$ and $c_1^2(X(7)) = 5$ by the similar argument as above.

Remark 24. The observation above that one more lantern relation could be found which allows a sequence of seven rational blowdowns to be performed on $K3\#2\overline{\mathbb{CP}}^2$ to acquire $X(7)$ rather six rational blowdowns could potentially have a deeper geometric meaning rather than merely constructing a smaller exotic 4-manifold. By the work of E. Hironaka [20], one can ‘read’ lantern relation from the planar line arrangement (Thm 1.2 in [20]). In our case the lantern relation corresponds to the triangle formed by the lines in the branch locus of the double branched covering description for the $K3\#2\overline{\mathbb{CP}}^2$. Interestingly, there are correspondingly seven triangles in generic arrangement of six lines which is the linear system $|6\tilde{L}|$ for our branch locus. It is tempting to postulate that one can find seven lantern relation in the global monodromy of $K3\#2\overline{\mathbb{CP}}^2$ and we have found them by mapping class group factorization calculus. The concrete interplay between the change in the branch locus of the Hyperelliptic Lefschetz fibration and its braid group monodromy in connection with the change in the topological structure of the Hyperelliptic Lefschetz fibration and its mapping class group monodromy is an interesting avenue to be investigated upon which was surveyed by I. Smith and D. Auroux in [4]. We will continue our investigation on this topic in an upcoming project [29].

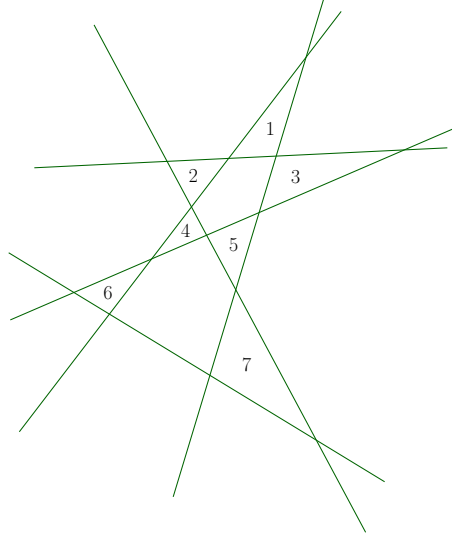


FIGURE 9. Seven Triangles on Branch Locus for $K3\#2\overline{\mathbb{CP}}^2$

6. FIBER SUM DECOMPOSABILITY AND DECOMPOSITION

In this section we prove the decomposability of $X(n)$ for $2 \leq n \leq 6$ and consider their possible decompositions under the genus 2 fiber sum.

Theorem 25 (Decomposability of $X(n)$ for $2 \leq n \leq 6$). *The genus 2 Lefschetz fibrations $X(n)$ for $2 \leq n \leq 6$ are all decomposable into nontrivial fiber sum of other genus 2 Lefschetz fibrations. Namely, $X(2)$ is isomorphic to an untwisted fiber*

sum of Matsumoto fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with Lefschetz fibration on $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Additionally, $X(3), X(4), X(5), X(6)$ are isomorphic to an untwisted fiber sum of Matsumoto fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ with $Z(1), Z(2), Z(3), Z(4)$ respectively.

Proof. As $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ portion of the monodromy can be blown down independently (not using the conjugation \xrightarrow{C}) by the above Lemma 11, it is easy to see that the untwisted fiber sum of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ having the positive relation $(\eta_1\delta\eta_2\eta_3)^2$ with $Z(m)$ having the positive relations of Lemma 11 for $0 \leq m \leq 4$ will give exotic copies $X(2), X(3), X(4), X(5), X(6)$ as indicated by the above monodromy factorizations of Lemma 12 which are the positive relations of $X(n)$ for $2 \leq n \leq 6$. \square

Theorem 26 (Unique decomposition of $X(2)$). *The genus 2 Lefschetz fibration $X(2)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (26, 2)$ must decompose under the genus 2 fiber sum having the indecomposable summands of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Each summands are determined up to diffeomorphism.*

Proof. Let us suppose $X(2)$ decomposes into two genus 2 Lefschetz fibrations $X(2) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are two possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to diffeomorphism.

First case is when the two reducible singular fibers distribute wholly to one of the summand (i.e. $s = (2, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (6, 2)$ and $Y(2)$ has $(n, s) = (20, 0)$. Then $Y(1)$ is diffeomorphic to Lefschetz fibrations $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is diffeomorphic to $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ by above proposition on characterization of genus 2 Lefschetz fibration with 20 irreducible singular fibers. Another possibility is when $Y(1)$ has $(n, s) = (16, 2)$ and $Y(2)$ has $(n, s) = (10, 0)$ and we know this is impossible by the remark 5.1 of [31], we know $(n, s) = (10, 0)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that these two decompositions are the only possibility for $s = (2, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (1, 1)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (8, 1)$ and $Y(2)$ has $(n, s) = (18, 1)$ then this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that this decomposition is the only possibility for $s = (1, 1)$ since $n + 2s \equiv 0 \pmod{10}$. \square

Proposition 27 (Decompositions of $X(3)$). *The genus 2 Lefschetz fibration $X(3)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (24, 3)$ must decompose under the genus 2 fiber sum having the summand of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(1) = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$ or the genus 2 Lefschetz fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and the genus 2*

Lefschetz fibration on $Z(0) = \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Each summands are determined up to diffeomorphism.

Proof. Let us suppose $X(3)$ decomposes into two genus 2 Lefschetz fibrations $X(3) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are two possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to diffeomorphism.

First case is when the three reducible singular fibers distribute wholly to one of the summand (i.e. $s = (3, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (4, 3)$ and $Y(2)$ has $(n, s) = (20, 0)$. Then $Y(1)$ is diffeomorphic to Lefschetz fibrations $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is diffeomorphic to $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ by above proposition on characterization of genus 2 Lefschetz fibration with 20 irreducible singular fibers. Note that this decomposition is the only possibility for $s = (3, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (1, 2)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (8, 1)$ and $Y(2)$ has $(n, s) = (16, 2)$ this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(1)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Another possibility is when $Y(1)$ has $(n, s) = (18, 1)$ then $Y(2)$ has $(n, s) = (6, 2)$ we know then $Y(1)$ is diffeomorphic to $\mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$ by above proposition on characterization of genus 2 Lefschetz fibration with 18 irreducible singular fibers and 1 reducible singular fiber and $Y(2)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31]. Note that these two decompositions are the only possibility for $s = (3, 1)$ since $n + 2s \equiv 0 \pmod{10}$. □

Remark 28. It is now known there exists a genus 2 Lefschetz fibration structure on $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ with seven singular fibers by the work of I. Baykur and M. Korkmaz [7] (In fact, they were able to show all the possible cases of minimal genus-2 Lefschetz fibrations whose total spaces are homeomorphic to simply-connected 4-manifold with $b_2^+ = 3$.)

Proposition 29 (Decompositions of $X(4)$). *The genus 2 Lefschetz fibration $X(4)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (22, 4)$ must decompose under genus 2 fiber sum having the summand of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(2) = \mathbb{CP}^2 \# 11\overline{\mathbb{CP}}^2$ or the genus 2 Lefschetz fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(1) = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$. Each summands are determined up to diffeomorphism except for the $Z(2)$ which is only determined up to homeomorphism.*

Proof. Let us suppose $X(4)$ decomposes into two genus 2 Lefschetz fibrations $X(4) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are three possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to homeomorphism.

First case is when the four reducible singular fibers distribute wholly to one of the summand (i.e. $s = (4, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (2, 4)$ and $Y(2)$ has $(n, s) = (20, 0)$. This is impossible as $N(2, 0) = \{7, 8\}$ (i.e. the minimal number of singular fibers in a genus 2 Lefschetz fibration

over \mathbb{S}^2 is 7 or 8 [27]) whereas $Y(1)$ has 6 singular fibers. Another possibility is when $Y(1)$ has $(n, s) = (12, 4)$ and $Y(2)$ has $(n, s) = (10, 0)$ and we know this is also impossible by the remark 5.1 of [31], as we know $(n, s) = (10, 0)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that these two decompositions are the only possibility for $s = (4, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (3, 1)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (14, 3)$ and $Y(2)$ has $(n, s) = (8, 1)$ this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Another possibility is when $Y(1)$ has $(n, s) = (4, 3)$ and $Y(2)$ has $(n, s) = (18, 1)$ then we know $Y(1)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is diffeomorphic to $\mathbb{CP}^2 \# 12\overline{\mathbb{CP}}^2$ by the above proposition on characterization of genus 2 Lefschetz fibration with 18 irreducible singular fibers and 1 reducible singular fiber. Note that these two decompositions are the only possibility for $s = (3, 1)$ since $n + 2s \equiv 0 \pmod{10}$.

Third case is when $s = (2, 2)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (6, 2)$ and $Y(2)$ has $(n, s) = (16, 2)$ we know then $Y(1)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is homeomorphic to $\mathbb{CP}^2 \# 11\overline{\mathbb{CP}}^2$. Note that this decomposition is the only possibility for $s = (2, 2)$ since $n + 2s \equiv 0 \pmod{10}$. □

Proposition 30 (Decompositions of $X(5)$). *The genus 2 Lefschetz fibration $X(5)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (20, 5)$ must decompose under genus 2 fiber sum having the summands of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(3) = \mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$ or the genus 2 Lefschetz fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(2) = \mathbb{CP}^2 \# 11\overline{\mathbb{CP}}^2$. The $Z(3)$ and $Z(2)$ genus 2 Lefschetz fibration summands are determined up to homeomorphism. The $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ genus 2 Lefschetz fibration summands are determined up to diffeomorphism.*

Proof. Let us suppose $X(5)$ decomposes into two genus 2 Lefschetz fibrations $X(5) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are three possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to homeomorphism.

First case is when the five reducible singular fibers distribute wholly to one of the summand (i.e. $s = (5, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (0, 5)$ and $Y(2)$ has $(n, s) = (20, 0)$. This is impossible as there is no hyperelliptic Lefschetz fibration over \mathbb{S}^2 with only reducible singular fibers (cf. [27]) whereas $Y(1)$ has 5 reducible singular fibers only. Another possibility is when $Y(1)$ has $(n, s) = (10, 5)$ and $Y(2)$ has $(n, s) = (10, 0)$. This is impossible by the remark 5.1 of [31], as we know $(n, s) = (10, 0)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that these two decompositions are the only possibility for $s = (5, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (4, 1)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (12, 4)$ and $Y(2)$ has $(n, s) = (8, 1)$ this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Another possibility is when $Y(1)$ has $(n, s) = (2, 4)$ and $Y(2)$ has $(n, s) = (18, 1)$ This is impossible as $N(2, 0) = \{7, 8\}$ (i.e. the minimal number of singular fibers in a genus 2 Lefschetz fibration over \mathbb{S}^2 is 7 or 8) [27] whereas $Y(1)$ has 6 singular fibers. Note that these two decompositions are the only possibility for $s = (4, 1)$ since $n + 2s \equiv 0 \pmod{10}$.

Third case is when $s = (2, 3)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (6, 2)$ and $Y(2)$ has $(n, s) = (14, 3)$ we know then $Y(1)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is homeomorphic to $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$. Another possibility is when $Y(1)$ has $(n, s) = (4, 3)$ and $Y(2)$ has $(n, s) = (16, 2)$ then we know $Y(1)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31] and $Y(2)$ is homeomorphic to $\mathbb{CP}^2 \# 11\overline{\mathbb{CP}}^2$. Note that these two decompositions are the only possibility for $s = (2, 3)$ since $n + 2s \equiv 0 \pmod{10}$. □

Proposition 31 (Decompositions of $X(6)$). *The genus 2 Lefschetz fibration $X(6)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (18, 6)$ must decompose under genus 2 fiber sum having the summand of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(4) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ or the genus 2 Lefschetz fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and the genus 2 Lefschetz fibration on $Z(3) = \mathbb{CP}^2 \# 10\overline{\mathbb{CP}}^2$. The $Z(4)$ and $Z(3)$ genus 2 Lefschetz fibration summands are determined up to homeomorphism. The $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}}^2$ and $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ genus 2 Lefschetz fibration summands are determined up to diffeomorphism.*

Proof. Let us suppose $X(6)$ decomposes into two genus 2 Lefschetz fibrations $X(2) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are four possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to homeomorphism.

First case is when the six reducible singular fibers distribute wholly to one of the summand (i.e. $s = (6, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (8, 6)$ and $Y(2)$ has $(n, s) = (10, 0)$. This is impossible by the remark 5.1 of [31], as we know $(n, s) = (10, 0)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration. Note that this decomposition is the only possibility for $s = (6, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (5, 1)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (0, 5)$ and $Y(2)$ has $(n, s) = (18, 1)$ this is impossible by the remark 5.1 of [31], as we know there is no hyperelliptic Lefschetz fibration over \mathbb{S}^2 with only reducible singular fibers [27] whereas $Y(1)$ has 5 reducible singular fibers only. Another possibility is when $Y(1)$ has $(n, s) = (10, 5)$ and $Y(2)$ has $(n, s) = (8, 1)$ this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that these two decompositions are the only possibility for $s = (5, 1)$ since $n + 2s \equiv 0 \pmod{10}$.

Third case is when $s = (4, 2)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (2, 4)$ then $Y(2)$ has $(n, s) = (16, 2)$ this is impossible as $N(2, 0) = \{7, 8\}$ (i.e. the minimal number of singular fibers in a genus 2 Lefschetz fibration over \mathbb{S}^2 is 7 or 8) [27] whereas $Y(1)$ has 6 singular fibers. Another possibility is when $Y(1)$ has $(n, s) = (12, 4)$ and $Y(2)$ has $(n, s) = (6, 2)$ then we know $Y(1)$ is homeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ and $Y(2)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}^2}$ by the proposition 4.1 [31]. Note that these two decompositions are the only possibility for $s = (4, 2)$ since $n + 2s \equiv 0 \pmod{10}$.

Fourth case is when $s = (3, 3)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (4, 3)$ and $Y(2)$ has $(n, s) = (14, 3)$ then we know $Y(1)$ is diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 3\overline{\mathbb{CP}^2}$ by the proposition 4.1 [31] and $Y(2)$ is homeomorphic to $\mathbb{CP}^2 \# 10\overline{\mathbb{CP}^2}$. Note that this decomposition is the only possibility for $s = (3, 3)$ since $n + 2s \equiv 0 \pmod{10}$. \square

Remark 32. Even though one can easily see indecomposability of $X(0)$ and $X(1)$ from non-minimality (cf. [37]) one can also prove $X(0)$ and $X(1)$ are indecomposable under the genus 2 fiber sum by the similar reasoning on the possible pairs of (n, s) for the summands.

As $X(0)$ has 30 irreducible singular fibers $(n, s) = (30, 0)$ if it were to decompose into two genus 2 Lefschetz fibrations $X(2) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations there is only one possible case of decomposition. Since $n + 2s \equiv 0 \pmod{10}$, without the loss of the generality $Y(1)$ has $(n, s) = (10, 0)$ and $Y(2)$ has $(n, s) = (20, 0)$ this is impossible by the remark 5.1 of [31], we know $(n, s) = (10, 0)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration.

Similarly for $X(1)$ which has $(n, s) = (28, 1)$ we can consider possible pairs of (n, s) for both $Y(1), Y(2)$. There are only two possible cases to consider namely when $Y(1)$ has $(n, s) = (8, 1)$ while $Y(2)$ has $(n, s) = (20, 0)$ and another possible case when $Y(1)$ has $(n, s) = (18, 1)$ while $Y(2)$ has $(n, s) = (10, 0)$. Both cases are impossible by the remark 5.1 of [31], we know $(n, s) = (10, 0)$ and $(n, s) = (8, 1)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration and thus such decomposition is impossible.

Remark 33. Similar reasoning on the possible pairs of (n, s) for the summands applies also to the Endo-Gurtas examples such as $Z(m)$ for $0 \leq m \leq 3$ to show indecomposability.

As $Z(0)$ has 20 irreducible singular fibers $(n, s) = (20, 0)$ if it were to decompose into two genus 2 Lefschetz fibrations $X(2) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations there is only one possible case of decomposition. Since $n + 2s \equiv 0 \pmod{10}$, without the loss of the generality $Y(1)$ has $(n, s) = (10, 0)$ and $Y(2)$ has $(n, s) = (10, 0)$ this is impossible by the remark 5.1 of [31], we know $(n, s) = (10, 0)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration.

Similarly for $Z(1)$ which has $(n, s) = (18, 1)$ we can consider possible pairs of (n, s) for both $Y(1), Y(2)$. There is only one possible case to consider namely when $Y(1)$ has $(n, s) = (8, 1)$ while $Y(2)$ has $(n, s) = (10, 0)$ whereas we know both are ruled out of existence by the remark 5.1 of [31].

As for $Z(2)$ there are only two possible cases to consider namely when $Y(1)$ has $(n, s) = (6, 2)$ while $Y(2)$ has $(n, s) = (10, 0)$ and another possible case when $Y(1)$ has $(n, s) = (8, 1)$ while $Y(2)$ has $(n, s) = (8, 1)$. Both cases are impossible as the remark 5.1 of [31], we know $(n, s) = (10, 0)$ and $(n, s) = (8, 1)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration and thus such decomposition is impossible.

Finally for the $Z(3)$, there are again only two possible cases to consider namely when $Y(1)$ has $(n, s) = (4, 3)$ while $Y(2)$ has $(n, s) = (10, 0)$ and another possible case when $Y(1)$ has $(n, s) = (16, 2)$ while $Y(2)$ has $(n, s) = (8, 1)$. Both cases are again impossible by the remark 5.1 of [31], we know $(n, s) = (10, 0)$ and $(n, s) = (8, 1)$ cannot occur as the pair of number of singular fibers for Lefschetz fibration and thus such decomposition is impossible.

Interestingly, it is impossible to rule out the decomposability of $Z(4)$ as suggested by Endo-Gurtas,

Proposition 34 (Decompositions of $Z(4)$). *$Z(4)$ which has n irreducible singular fibers and s reducible singular fibers pair $(n, s) = (12, 4)$ if it were to decompose it must decompose under genus 2 fiber sum having the indecomposable summands of Matsumoto's fibration on $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$. The summands are determined up to diffeomorphism.*

Proof. Let us suppose $Z(4)$ decomposes into two genus 2 Lefschetz fibrations $X(2) = Y(1) \# Y(2)$ where both $Y(1), Y(2)$ are relatively minimal genus 2 Lefschetz fibrations. There are three possible cases to consider for the distribution of reducible singular fibers and hence determine the possible decompositions up to diffeomorphism.

First case is when the four reducible singular fibers distribute wholly to one of the summand (i.e. $s = (4, 0)$) where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (2, 4)$ and $Y(2)$ has $(n, s) = (10, 0)$. This is impossible as $N(2, 0) = \{7, 8\}$ (i.e. the minimal number of singular fibers in a genus 2 Lefschetz fibration over \mathbb{S}^2 is 7 or 8) [27] whereas $Y(1)$ has 6 singular fibers. It is also impossible by the remark 5.1 of [31], as we know $(n, s) = (10, 0)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that this is the only possible decomposition case to consider for $s = (4, 0)$ since $n + 2s \equiv 0 \pmod{10}$.

Second case is when $s = (3, 1)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (4, 3)$ and $Y(2)$ has $(n, s) = (8, 1)$ this is impossible by the remark 5.1 of [31], as we know $(n, s) = (8, 1)$ (the (n, s) pair for $Y(2)$) cannot occur as the pair of number of singular fibers for genus 2 Lefschetz fibration. Note that this is the only possible decomposition case to consider for $s = (3, 1)$ since $n + 2s \equiv 0 \pmod{10}$.

Third case is when $s = (2, 2)$, where without the loss of generality, we can assume $Y(1)$ has $(n, s) = (6, 2)$ and $Y(2)$ has $(n, s) = (6, 2)$ we know then $Y(1)$ and $Y(2)$ must be diffeomorphic to genus 2 Lefschetz fibration $\mathbb{S}^2 \times \mathbb{T}^2 \# 4\overline{\mathbb{CP}}^2$ by the proposition 4.1 [31]. \square

As it is still not known whether or not $Z(4)$ in our article or E in Endo-Gurtas are actually decomposable into the two genus 2 Lefschetz fibrations to begin with

this decomposition result alone does not fully answer the question asked by Endo-Gurtas [13].

ACKNOWLEDGMENTS

I am grateful to Anar Akhmedov and Refik İnanç Baykur for suggesting this problem and for many useful discussions and ideas. I am also grateful to Tian-Jun Li, András I. Stipsicz and Chuen-Ming Michael Wong for helpful conversations.

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